NORTH SYDNEY BOYS HIGH SCHOOL

2011 HSC ASSESSMENT TASK 2

Mathematics

General Instructions

- Working time – 55 minutes
- Write on the lined paper in the booklet provided
- Write using blue or black pen
- Board approved calculators may be used
- All necessary working should be shown in every question
- Each new question is to be started on a **new page.**
- Attempt all questions

Class Teacher:
(Please tick or highlight)

- O Mr Berry
- O Ms Ziaziaris
- O Mr Law
- O Mr Weiss
- O Mr Lam
- O Mr Ireland
- O Mr Fletcher
- O Ms Collins/Mr Rezcallah

Student Number:___________________________

(To be used by the exam markers only.)

Question No	1	2	3	4	Total	Total %
Mark	/22	/18	/12	/4	/56	/100

Question 1 (22 marks)

a) Find.

i. $\int \frac{7}{x^2} dx$ 2

ii. $\int \frac{5x^2 + x^3}{x^2} dx$ 2

iii. $\int \sqrt{x-1}\ dx$ 2

b) Evaluate.

i. $\int_{-8}^{8} x^3\, dx$ 1

ii. $\int_{2}^{6} (x+1)(5x-2)\, dx$ 2

iii. $\int_{1}^{4} x^2\sqrt{x}\, dx$ 2

c)

i. Sketch the graph of $y = 16 - x^2$. 2

ii. Approximate the area under this graph between x=1 and x=3 and the x axis using the trapezoidal rule with three function values. 3

iii. Integrate the same section of the graph and find an exact value. 2

iv. Explain why the areas to part ii and iii are different. 1

d) Find the value of k.

$$\int_{2}^{k} (x+3)\, dx = 30$$ 3

Question 2 (18 marks)   Start A New Page

a) The quadratic $2x^2 + 5x + 1 = 0$ has roots α and β.
Without finding the roots, evaluate:

i. α + β   1

ii. α β   1

iii. $\alpha^2 + \beta^2$ 2

iv. $\frac{1}{\alpha} + \frac{1}{\beta}$ 1

b) Given the roots of a quadratic equation are: $\alpha = 3 + \sqrt{5}$ and $\beta = 3 - \sqrt{5}$. Write the equation in the form $ax^2 + bx + c = 0$.   3

c) Solve: 3

$$x^4 - 17x^2 + 16 = 0$$

d) Find the values of A, B and C   3

$$5x^2 + 6x - 7 \equiv A(x-2)^2 + B(x-2) + C$$

e) Given the quadratic function

$$y = 2x^2 + kx + 8$$

i. For what values of *k* does the function pass through the point (2,0)?   1

ii. For what values of *k* is the equation positive definite.   2

iii. Can the function be negative definite? Why?   1

Question 3 (12 marks) Start A New Page

a) Given the parabola.

$$y = (x-1)^2 - 3$$

Find:

i. the vertex 1
ii. the focus 2
iii. the equation of the directrix 1
iv. Sketch the parabola, indicating clearly the directrix, vertex and focus. 2

b) Find the equation of the locus of a point, P, that is equidistant from two points A(4, 6) and B(8, 10) 2

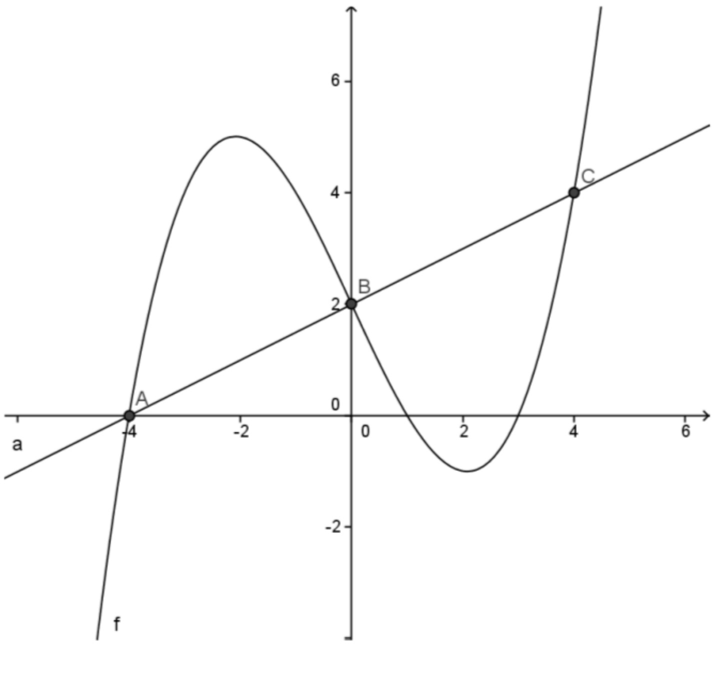
c)

i) Find the locus of a point P(x,y) which moves so its distance from point A (-3,3) is twice the distance from the point B (2, -9). 2
ii) Describe this locus geometrically. 2

Question 4 (4 marks) Start A New Page

The curves $y = \frac{1}{6}(x-3)(x-1)(x+4)$ and $y = 2x + 2$ intersect at the points A(-4,0), B(0,2) and C(4,4) as shown in the diagram. 4

Calculate the area enclosed between the two curves.



Suggested Solutions

Question 1

(a) i. (2 marks)

$$\int \frac{7}{x^2}\,dx = \int 7x^{-2}\,dx$$
$$= -7x^{-1} + C$$

ii. (2 marks)

$$\int \frac{5x^2+x^3}{x^2}\,dx = \int \left(\frac{5\cancel{x^2}}{\cancel{x^2}} + \frac{x^{\cancel{3}}}{\cancel{x^2}}\right)\,dx$$
$$= \int (5+x)\,dx$$
$$= 5x + \frac{1}{2}x^2 + C$$

iii. (2 marks)

$$\int \sqrt{x-1}\,dx = \int (x-1)^{\frac{1}{2}}\,dx$$
$$= \frac{2}{3}(x-1)^{\frac{3}{2}} + C$$

(b) i. (1 mark)

$$\int_{-8}^{8} x^3\,dx = 0$$

(integral of odd function over balanced interval centred around the origin)

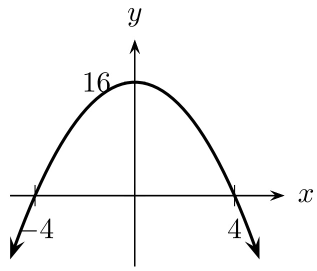
ii. (2 marks)

$$\int_2^6 (x+1)(5x-2)\,dx$$
$$= \int_2^6 \left(5x^2+3x-2\right)\,dx$$
$$= \left[\frac{5x^3}{3} + \frac{3x^2}{2} - 2x\right]_2^6$$
$$= \frac{5}{3}\left(6^3-2^2\right) + \frac{3}{2}\left(6^2-2^2\right)$$
$$- 2(6-2)$$
$$= \frac{1\,160}{3}$$

iii. (2 marks)

$$\int_1^4 x^2\sqrt{x}\,dx = \int_1^4 x^2 \times x^{\frac{1}{2}}\,dx$$
$$= \int_1^4 x^{\frac{5}{2}}\,dx$$
$$= \left[\frac{2}{7}x^{\frac{7}{2}}\right]_1^4$$
$$= \frac{2}{7}\left(4^{\frac{7}{2}}-1\right)$$
$$= \frac{254}{7}$$

(c) i. (2 marks)



ii. (3 marks)

x	1	2	3
$16-x^2$	15	12	7

$$A \approx \frac{h}{2}\left(y_1 + 2\sum y_{\text{middle}} + y_\ell\right)$$
$$= \frac{1}{2}\left(15 + 2(12) + 7\right)$$
$$= 23$$

iii. (2 marks)

$$\int_1^3 16 - x^2\,dx$$
$$= \left[16x - \frac{1}{3}x^3\right]_1^3$$
$$= 16(3-1) - \frac{1}{3}(3^3-1)$$
$$= \frac{70}{3} \approx 23.33$$

iv. (1 mark)
The areas are different as the trapezoidal rule only gives an approximation to the exact area.

(d) (3 marks)

$$\int_2^k (x+3)\,dx = \left[\frac{1}{2}x^2 + 3x\right]_2^k$$
$$= \frac{1}{2}(k^2 - 2^2) + 3(k-2)$$
$$= 30$$
$$\therefore k^2 - 4 + 6(k-2) = 60$$
$$k^2 + 6k - 16 = 60$$
$$k^2 + 6k + 9 = 76 + 9 = 85$$
$$(k+3)^2 = 85$$
$$k + 3 = \pm\sqrt{85}$$
$$\therefore k = -3 \pm \sqrt{85}$$

Question 2

(a) $2x^2 + 5x + 1 = 0$

i. (1 mark)

$$\alpha + \beta = -\frac{b}{a} = -\frac{5}{2}$$

ii. (1 mark)

$$\alpha\beta = \frac{c}{a} = \frac{1}{2}$$

iii. (2 marks)

$$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - \alpha\beta$$
$$= \frac{25}{4} - \frac{1}{2} = \frac{23}{4}$$

iv. (1 mark)

$$\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha+\beta}{\alpha\beta} = \frac{-\frac{5}{2}}{\frac{1}{2}} = -5$$

(b) (3 marks)

$$x^2 - (\alpha + \beta)x + (\alpha\beta) = 0$$
$$\alpha = 3 + \sqrt{5} \qquad \beta = 3 - \sqrt{5}$$
$$\therefore x^2 - 6x + 4 = 0$$

(c) (3 marks)

$$x^4 - 17x^2 + 16 = 0$$

Let $m = x^2$,

$$m^2 - 17m + 16 = 0$$
$$(m-16)(m-1) = 0$$
$$m = 1, 16 \therefore x^2 = 1, 16$$
$$\therefore x = \pm 1, \pm 4$$

(d) (3 marks)

$5x^2 + 6x - 7 \equiv A(x-2)^2 + B(x-2) + C$

By inspection, $A = 5$. Letting $x = 2$,

$$5(2^2) + 6(2) - 7 = C$$
$$C = 20 + 12 - 7 = 25$$

Let $x = 1$,

$$5 + 6 - 7 = 5 - B + 25$$
$$4 = 30 - B$$
$$\therefore B = 26$$

Hence $A = 5$, $B = 26$, $C = 25$.

(e) $y = 2x^2 + kx + 8$

i. (1 mark)

$$x = 2 \qquad y = 0$$
$$0 = 2(2^2) + 2k + 8$$
$$2k + 16 = 0$$
$$\therefore k = -8$$

ii. (2 marks)

$$\Delta = b^2 - 4ac > 0$$
$$k^2 - 4(2)(8) > 0 k^2 - 64 > 0$$
$$(k-8)(k+8) > 0$$
$$\therefore k < -8 \qquad k > 8$$

iii. (1 mark)
No, as the coefficient of x^2 is positive.

Question 3

(a) $y = (x-1)^2 - 3$

i. (1 mark)

$$V(1, -3)$$

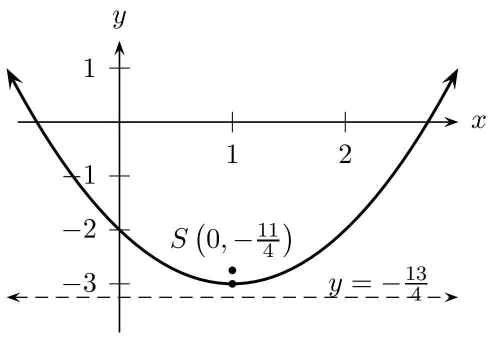
ii. (2 marks)

$$(x-1)^2 = 4 \times \frac{1}{4}(y+3)$$
$$\therefore a = \frac{1}{4}$$
$$\therefore S\left(1, -\frac{11}{4}\right)$$

iii. (1 mark)

$$y = -3 - \frac{1}{4} = -\frac{13}{4}$$

iv. (2 marks)



(b) (2 marks)

$$P(x,y) \qquad A(4,6) \qquad B(8,10)$$
$$PA = \sqrt{(x-4)^2 + (y-6)^2}$$
$$PB = \sqrt{(x-8)^2 + (y-10)^2}$$

As $PA = PB$,

$$(x-4)^2 + (y-6)^2 = (x-8)^2 + (y-10)^2$$
$$(x-4)^2 - (x-8)^2 = (y-10)^2 + (y-6)^2$$
$$\Big[(x-4)-(x-8)\Big]\Big[(x-4)+(x-8)\Big]$$
$$= \Big[(y-10)-(y-6)\Big]\Big[(y-10)+(y-6)\Big]$$
$$\not{4}(2x-12) = -\not{4}(2y-16)$$
$$x - 6 = -(y-8)$$
$$x + y = 14$$

(c) i. (2 marks)

$$P(x,y) \quad A(-3,3) \quad B(2,-9)$$
$$PA = \sqrt{(x+3)^2 + (y-3)^2}$$
$$PB = \sqrt{(x-2)^2 + (y+9)^2}$$
$$PA = 2PB$$
$$(x+3)^2 + (y-3)^2 = 4(x-2)^2 + 4(y+9)^2$$
$$4(x-2)^2 - (x+3)^2 = (y-3)^2 - 4(y+9)^2$$
$$\Big[2(x-2)-(x+3)\Big]\Big[2(x-2)+(x+3)\Big]$$
$$= \Big[(y-3)-2(y+9)\Big]\Big[(y-3)+2(y+9)\Big]$$
$$(x-7)(3x-1) = (-y-21)(3y+15)$$
$$3x^2 - 22x + 7 = -3y^2 - 78y - 315$$
$$3x^2 + 3y^2 - 22x + 78y = -322$$
$$x^2 - \frac{22}{3}x + y^2 + 26y = -\frac{322}{3}$$
$$x^2 - \frac{22}{3}x + \left(\frac{11}{3}\right)^2 + y^2 + 26y + 13^2$$
$$= -\frac{322}{3} + \left(\frac{11}{3}\right)^2 + 169$$
$$\left(x - \frac{11}{3}\right)^2 + (y+13)^2 = \frac{676}{9} = \left(\frac{24}{3}\right)^2$$

ii. (2 marks)
Circle with centre $\left(\frac{11}{3}, -13\right)$ and radius $\frac{24}{3}$

Question 4

$$y = \frac{1}{6}(x-3)(x-1)(x+4)$$
$$= \frac{1}{6}\left(x^2 - 4x + 3\right)(x+4)$$
$$= \frac{1}{6}\left(x^3 - 13x + 12\right)$$
$$A = \int_{-4}^{0} \frac{1}{6}\left(x^3 - 13x + 12\right) - (2x+2)\,dx$$
$$+ \int_{0}^{4} (2x+2) - \frac{1}{6}\left(x^3 - 13x + 12\right)\,dx$$
$$= 2\int_0^4 (2x+2) - \frac{1}{6}x^3 + \frac{13}{6}x - 2\,dx$$
$$= 2\int_0^4 -\frac{1}{6}x^3 + \frac{25}{6}x\,dx$$
$$= 2\left[-\frac{1}{24}x^4 + \frac{25}{12}x^2\right]_0^4$$
$$= 2\left(-\frac{1}{24}\times 4^4 + \frac{25}{12}\times 4^2\right) = \frac{136}{3}$$